

Approximate homotopy series solutions of perturbed PDEs via approximate symmetry method

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Abstract: We show that the two couple equations derived by approximate symmetry method and approximate homotopy symmetry method are connected by a transformation for the perturbed PDEs. Consequently, approximate homotopy series solutions can be obtained by acting the transformation on the known solutions by approximate symmetry method. Applications to the Cahn-Hilliard equation illustrate the effectiveness of the transformation.
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1 Introduction

Group theoretical methods based on local and nonlocal symmetries provide remarkable techniques to analyze partial differential equations (PDEs) such as constructing group invariant solutions, studying integrability etc., and also build a ladder to explore profound understandings of the physics of the underlying problems such as detecting conservation laws [1,2]. The prerequisite of achieving these goals is that the PDEs under study possess affluent symmetries, nevertheless, for the perturbed PDEs (containing a small perturbed parameter), the classical symmetry method losses its superiorities because the small parameter disturbs the symmetry group properties of the unperturbed equation, even makes the studied PDEs hold few symmetries.

In view of the drawbacks of classical symmetry method for conducting this type of equations, the integration of perturbation analysis and symmetry group theory makes two well-known approximate symmetry methods (ASM) emerge [3,4]. The originators of the two approaches are Baikov *et.al.* [3] and Fushchich and Shtelen [4] who employ the standard perturbation techniques on the infinitesimal operators and dependent variables respectively. The comparisons of the two ASM which are illustrated by several specific equations may be found in [5,6]. Quite recently, the extended combination of homotopy conception with perturbation technique and symmetry method produces approximate homotopy symmetry method (AHSM), which is suitable for studying the problems which don't contain small parameters [7].

The distinct merit of AHSM is that the accuracy of approximate homotopy series solutions can be controlled by adjusting the convergence-control parameter, however, it is at

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the expense of the greater computation than ASM [8]. On the other hand, we find that the frameworks of ASM and AHSM for the perturbed PDEs are quite similar except for the different studied objects (see subsection 2.1 below). Hence, a natural question occurs: what is the relationship between ASM and AHSM, does there exist a transformation or others to link them? In [8], the authors study this problem which is exemplified by a class of perturbed nonlinear wave equations and show that first-order coupled equations and higher-order similarity reduced equations are connected by two scaling transformations respectively. However, the general results for the connections between ASM and AHSM for general perturbed PDEs are still open.

The purpose of our paper is to give a positive answer about this question. We show that the two couple equations obtained by ASM and AHSM are equivalent under a transformation. Based on this transformation, one can construct approximate homotopy series solutions directly from the known approximate solutions by ASM without performing AHSM for the governing equations.

The outline of the paper is as follows: In section 2, we briefly review some basic notions and give the main results. Section 3 concentrate on the applications of the results to the Cahn-Hilliard equation. The last section contains conclusion and discussion of our results.

2 Basic notions and principles

In this section, we first briefly review the main ideas of ASM and AHSM, and then give the results about the relationship between the two approaches.

2.1 Basic notions

We take the following k th-order perturbed nonlinear PDE

$$E = E_0(x, u, u^{(1)}, \dots, u^{(k)}) + \epsilon E_1(x, u, u^{(1)}, \dots, u^{(k)}) = 0 \quad (1)$$

into account to illustrate the main ideas of ASM and AHSM, throughout the section, with independent variables $x = (x_1, \dots, x_n) (n \geq 2)$, a single dependent variable u , $u^{(j)} = \partial^j u / \partial x_1^{j_1} \dots \partial x_n^{j_n}$ with $j_1 + \dots + j_n = j$ ($j, j_1, \dots, j_n = 1, \dots, k$) is a collection of all j th-order partial derivatives, ϵ is a small perturbed parameter. Note that we only use one variable u to keep the notation simple, the generalization to systems being fairly obvious. Also $E_i = E_i(x, u, u^{(1)}, \dots, u^{(k)}) (i = 0, 1)$ if no special notes are added.

Let's first recall the definition of classical symmetry. Consider the infinitesimal generator

$$X = \sum_{i=1}^n \xi^i(x, u_0, u_1 \dots, u^{(k)}) \partial_{x_i} + \eta(x, u_0, u_1 \dots, u^{(k)}) \partial_u, \quad (2)$$

where ξ^i and η are smooth functions of their arguments. Let $X^{(k)}$ be the k th-prolongation of the generator X calculated by the well-known prolongation formulae [1, 2].

Definition 1. (Symmetry or exact symmetry [1, 2]) The Lie symmetry X of the form (2) is admitted by Eq.(1) if and only if

$$X^{(k)} E|_{E=0} = 0,$$

where the notation $|_{E=0}$ means evaluated on the solution manifold of $E = 0$.

The ASM originated by Fushchich and Shtelen employs a perturbation of dependent variable and then approximate symmetry of original equation is defined to the exact symmetry of the system corresponding to each order in the small parameter.

Specifically, expanding the dependent variable with respect to the small parameter ϵ yields

$$u = \sum_{l=0}^{\infty} \epsilon^l u_l,$$

where $u_l = u_l(x)$ ($l = 0, 1, 2, \dots$) stand for unknown functions throughout the paper, then substituting it into Eq.(1) and vanishing the coefficients of all different powers of ϵ , one obtains a coupled system

$$\begin{aligned} \epsilon^0 : \quad & E_0|_{\epsilon=0} = 0, \\ \epsilon^i : \quad & \left[\frac{\partial^i E_0}{\partial \epsilon^i} + i \frac{\partial^{i-1} E_1}{\partial \epsilon^{i-1}} \right]_{|\epsilon=0} = 0, \quad i = 1, 2, \dots \end{aligned} \quad (3)$$

Definition 2. (Approximate symmetry [4]) The i th order approximate symmetry of Eq.(1) is defined to the exact symmetry of the first $i + 1$ equations in Eq.(3).

For approximate homotopy symmetry of Eq.(1), one generally consider the following simple homotopy model

$$H(u, q) = (1 - \theta q)E_0 + q\epsilon(1 - \theta)E_1 = 0, \quad (4)$$

where, hereinafter, $q \in [0, 1]$ is an embedding homotopy parameter and θ denotes the convergence-control parameter. The above homotopy model has the property $H(u, 0) = E_0$, $H(u, 1) = E$.

Assuming Eq.(1) has the homotopy series solutions in the form

$$u = \sum_{l=0}^{\infty} q^l u_l, \quad (5)$$

where u_l solve

$$\begin{aligned} q^0 : \quad & E_0|_{q=0} = 0, \\ q^i : \quad & \left[\frac{\partial^i E_0}{\partial q^i} - i\theta \frac{\partial^{i-1} E_0}{\partial q^{i-1}} + i\epsilon(1 - \theta) \frac{\partial^{i-1} E_1}{\partial q^{i-1}} \right]_{|q=0} = 0, \quad i = 1, 2, \dots \end{aligned} \quad (6)$$

Equivalently, we rearrange Eq.(6) by eliminating $\partial^{i-1} E_0 / \partial q^{i-1}$ in the $(i + 1)$ th equation by means of the first i equations and transform it into the following form

$$\begin{aligned} q^0 : \quad & E_0|_{q=0} = 0, \\ q^i : \quad & \left[\frac{\partial^i E_0}{\partial q^i} + \epsilon(1 - \theta) \sum_{k=0}^{i-1} \theta^{i-1-k} \frac{i!}{k!} \frac{\partial^k E_1}{\partial q^k} \right]_{|q=0} = 0, \quad i = 1, 2, \dots \end{aligned} \quad (7)$$

Definition 3. (Approximate homotopy symmetry [7]) The i th order approximate homotopy symmetry of Eq.(1) associated with homotopy model (4) corresponds to the exact symmetry of the first $i + 1$ equations in Eq.(6) or Eq.(7).

Obviously, the first equation of Eq.(3) is the same as the first one of Eq.(7). When $\theta = 0$, Eq.(3) is equivalent to Eq.(7) under the scaling transformation $u_i = \epsilon^i \tilde{u}_i$ ($i = 0, 1, 2, \dots$) owing to $(\partial^i E_0 / \partial q^i)|_{q=0}$ is linear about u_i and its derivatives $u_i^{(\lambda)}$ (see Lemma 1 below). However, the parameter θ is used to adjust the convergence of homotopy series solution and may not be zero, thus it is the intention below to search for the connections under the condition $\theta \neq 0$.

2.2 Main results

In what follows, we first prove two basic lemmas and then establish the connections between Eq.(3) and Eq.(7).

Lemma 1. $(\partial^n E_0 / \partial q^n)|_{q=0}$ is linear about u_n and its derivatives $u_n^{(\lambda)}$, where $n(\geq 1)$ are positive integers and $\lambda = 0, 1, \dots, k$.

Proof: We take $u^{(0)}(=u), u^{(1)}, \dots, u^{(k)}$ of E_0 as independent arguments and expand u as (5), then by means of the generalized Faa di Bruno's formula [11], n th-order derivative of E_0 with respect to q is

$$\begin{aligned} \frac{\partial^n E_0}{\partial q^n} &= \sum_0 \sum_1 \cdots \sum_n \frac{n!}{\prod_{i=1}^n (i!)^{r_i} \prod_{i=1}^n \prod_{j=1}^{k+1} p_{ij}!} \times \frac{\partial^n E_0}{\partial [u^{(0)}]^{p_1} \partial [u^{(1)}]^{p_2} \cdots \partial [u^{(k)}]^{p_{k+1}}} \\ &\quad \times \prod_{i=1}^n \left(\frac{\partial^i u^{(0)}}{\partial q^i} \right)^{p_{i1}} \left(\frac{\partial^i u^{(1)}}{\partial q^i} \right)^{p_{i2}} \cdots \left(\frac{\partial^i u^{(k)}}{\partial q^i} \right)^{p_{i(k+1)}}, \end{aligned} \quad (8)$$

where the respective sums are over all nonnegative integer solutions of the Diophantine equations, as follows

$$\begin{aligned} \sum_0 : r_1 + 2r_2 + \cdots + nr_n &= n, \\ \sum_m : p_{m1} + p_{m2} + \cdots + p_{m(k+1)} &= r_m, \quad m = 1, 2, \dots, n, \\ r &= p_1 + p_2 + \cdots + p_{k+1} = r_1 + r_2 + \cdots + r_n, \\ p_j &= p_{1j} + p_{2j} + \cdots + p_{nj}, \quad j = 1, 2, \dots, k+1, \end{aligned} \quad (9)$$

and

$$\frac{\partial^i u^{(\lambda)}}{\partial q^i} = \sum_{l=i}^{\infty} \frac{l!}{(l-i)!} q^{l-i} u_l^{(\lambda)}, \quad \lambda = 0, 1, \dots, k. \quad (10)$$

Obviously, when $q = 0$, all terms in Eq.(10) vanish except for $u_i^{(\lambda)}$. Similarly, Eq.(8) with

$q = 0$ becomes

$$\begin{aligned} \frac{\partial^n E_0}{\partial q^n} \Big|_{q=0} &= \sum_0 \sum_1 \cdots \sum_n \frac{n!}{\prod_{i=1}^n \prod_{j=1}^{k+1} p_{ij}!} \times \left[\frac{\partial^r E_0}{\partial[u^{(0)}]^{p_1} \partial[u^{(1)}]^{p_2} \cdots \partial[u^{(k)}]^{p_{k+1}}} \right] \Big|_{q=0} \\ &\times \prod_{i=1}^n \left[u_i^{(0)} \right]^{p_{i1}} \left[u_i^{(1)} \right]^{p_{i2}} \cdots \left[u_i^{(k)} \right]^{p_{i(k+1)}}. \end{aligned} \quad (11)$$

In Eq.(11), only when $i = n$, the terms containing u_n (or its derivatives) occur. At the moment, the solution of the first equation in Eq.(9) $r_n = 1, r_i = 0 (i \neq n)$ generates the terms uniquely, then other equations give $r = 1$ and $p_i = 1, p_s = 0 (s \neq i)$ and $p_{ni} = 1, p_{ns} = 0 (s \neq i)$. Hence, the arguments u_n and its derivatives only occur in the following terms in Eq.(11)

$$\frac{\partial E_0}{\partial u^{(0)}} \Big|_{q=0} u_n^{(0)}, \quad \frac{\partial E_0}{\partial u^{(1)}} \Big|_{q=0} u_n^{(1)}, \quad \dots, \quad \frac{\partial E_0}{\partial u^{(k)}} \Big|_{q=0} u_n^{(k)},$$

which are linear about u_n and its derivatives since $(\partial E_0 / \partial u^{(i)})|_{q=0} (i = 0, 1, \dots, k)$ are only functions of x, u_0 and their derivatives. Meanwhile, all other terms with $q = 0$ are only functions of $x, u, u_1, \dots, u_{n-1}$ and their derivatives, thus Lemma 1 follows. The proof ends. \square

Lemma 2. Substituting the transformation

$$u_0^{(\lambda)} = \hat{u}_0^{(\lambda)}, \quad u_l^{(\lambda)} = \sum_{j=0}^{l-1} \binom{l-1}{j} \theta^j \hat{u}_{l-j}^{(\lambda)}, \quad l = 1, 2, \dots, \lambda = 0, 1, \dots, k, \quad (12)$$

into $(\partial^n E_0 / \partial q^n)|_{q=0}$, after deleting the symbol $\hat{\cdot}$, $[\partial^n E_0(\theta : u, \dots, u^{(k)}) / \partial q^n]|_{q=0}$ is equal to

$$\left[\frac{\partial^n E_0}{\partial q^n} + \sum_{i=2}^n \frac{n!}{(i-1)!} \binom{n-1}{i-2} \theta^{n-i+1} \frac{\partial^{i-1} E_0}{\partial q^{i-1}} \right] \Big|_{q=0}, \quad (13)$$

where $[\partial^n E_0(\theta : u, \dots, u^{(k)}) / \partial q^n]|_{q=0}$ denotes the transformed expression via (12) and $(\partial^i E_0 / \partial q^i)|_{q=0} (i = 2, 3, \dots, n)$ stand for the untransformed ones.

Proof: The main idea of the proof is to regard $[\partial^n E_0(\theta : u, \dots, u^{(k)}) / \partial q^n]|_{q=0}$ as a polynomial of θ and search for the coefficients of different degree of θ in it.

Specifically, transformation (12) makes Eq.(11) become

$$\begin{aligned} \frac{\partial^n E_0}{\partial q^n} \Big|_{q=0} &= \sum_0 \sum_1 \cdots \sum_n \frac{n!}{\prod_{i=1}^n \prod_{j=1}^{k+1} p_{ij}!} \times \left[\frac{\partial^r E_0}{\partial[u^{(0)}]^{p_1} \partial[u^{(1)}]^{p_2} \cdots \partial[u^{(k)}]^{p_{k+1}}} \right] \Big|_{q=0} \\ &\times \prod_{i=1}^n \left[\sum_{j=0}^{i-1} \binom{i-1}{j} \theta^j \hat{u}_{i-j}^{(0)} \right]^{p_{i1}} \cdots \left[\sum_{j=0}^{i-1} \binom{i-1}{j} \theta^j \hat{u}_{i-j}^{(k)} \right]^{p_{i(k+1)}} \\ &= \sum_0 \sum_1 \cdots \sum_n \frac{n!}{\prod_{i=1}^n \prod_{j=1}^{k+1} p_{ij}!} \times \left[\frac{\partial^r E_0}{\partial[u^{(0)}]^{p_1} \partial[u^{(1)}]^{p_2} \cdots \partial[u^{(k)}]^{p_{k+1}}} \right] \Big|_{q=0} \\ &\times [\hat{u}_1^{(0)}]^{p_{11}} \cdots [\hat{u}_1^{(k)}]^{p_{1(k+1)}} [\hat{u}_2^{(0)} + \theta \hat{u}_1^{(0)}]^{p_{21}} \cdots [\hat{u}_2^{(k)} + \theta \hat{u}_1^{(k)}]^{p_{2(k+1)}} \times \cdots \\ &\times [\hat{u}_n^{(0)} + \cdots + \theta^{n-1} \hat{u}_1^{(0)}]^{p_{n1}} \cdots [\hat{u}_n^{(k)} + \cdots + \theta^{n-1} \hat{u}_1^{(k)}]^{p_{n(k+1)}}. \end{aligned} \quad (14)$$

In particular,

$$\begin{aligned}
\left[\sum_{j=0}^{i-1} \binom{i-1}{j} \theta^j \widehat{u}_{i-j}^{(\lambda)} \right]^{p_{i\nu}} &= \sum \frac{p_{i\nu}! [\widehat{u}_i^{(\lambda)}]^{a_{\nu 0}}}{a_{\nu 0}! \dots a_{\nu(i-1)}!} \left[(i-1) \theta \widehat{u}_{i-1}^{(\lambda)} \right]^{a_{\nu 1}} \dots \left[\theta^{i-1} \widehat{u}_1^{(\lambda)} \right]^{a_{\nu(i-1)}} \\
&= \sum \frac{p_{i\nu}! [\widehat{u}_i^{(\lambda)}]^{a_{\nu 0}}}{a_{\nu 0}! \dots a_{\nu(i-1)}!} \left[(i-1) \widehat{u}_{i-1}^{(\lambda)} \right]^{a_{\nu 1}} \dots \left[\widehat{u}_1^{(\lambda)} \right]^{a_{\nu(i-1)}} \\
&\quad \times \theta^{a_{\nu 1} + 2a_{\nu 2} + \dots + (i-1)a_{\nu(i-1)}}, \tag{15}
\end{aligned}$$

where $\lambda = 0, 1, \dots, k$ and the sum is over all nonnegative integer solutions of the Diophantine equation $a_{\nu 0} + a_{\nu 1} + \dots + a_{\nu(i-1)} = p_{i\nu}$.

For Eq.(14), we first consider the terms without containing parameter θ . That is to say, let $\theta = 0$ in Eq.(14), we have

$$\begin{aligned}
\frac{\partial^n E_0}{\partial q^n} \Big|_{q=0} &= \sum_0 \sum_1 \dots \sum_n \frac{n!}{\prod_{i=1}^n \prod_{j=1}^{k+1} p_{ij}!} \times \left[\frac{\partial^r E_0}{\partial [\widehat{u}^{(0)}]^{p_1} \partial [\widehat{u}^{(1)}]^{p_2} \dots \partial [\widehat{u}^{(k)}]^{p_{k+1}}} \right] \Big|_{q=0} \\
&\quad \times \prod_{i=1}^n \left[\widehat{u}_i^{(0)} \right]^{p_{i1}} \left[\widehat{u}_i^{(1)} \right]^{p_{i2}} \dots \left[\widehat{u}_i^{(k)} \right]^{p_{i(k+1)}}, \tag{16}
\end{aligned}$$

which has the same form as Eq.(11).

Secondly, we look for the terms only involving θ in Eq.(14). The general formula of this type of term appears as follows

$$\prod_{j=0}^k [\widehat{u}_1^{(j)}]^{p_{1(j+1)}} \times \dots \times \binom{p_{m(j+1)}}{1} [\widehat{u}_m^{(j)}]^{p_{m(j+1)}-1} \theta \widehat{u}_{m-1}^{(j)} \times \dots \times [\widehat{u}_n^{(j)}]^{p_{n(j+1)}}, \quad m = 2, 3, \dots$$

At this time, the parameters satisfy

$$\begin{aligned}
\sum_0 &: r_1 + \dots + (m-1)(r_{m-1} + 1) + m(r_m - 1) + \dots + n r_n = n - 1, \\
\sum_{m-1} &: p_{(m-1)1} + p_{(m-1)2} + \dots + p_{(m-1)(k+1)} = r_{m-1} + 1, \quad m = 2, \dots, n, \\
\sum_m &: p_{m1} + p_{m2} + \dots + p_{m(k+1)} = r_m - 1, \quad m = 2, \dots, n, \\
\sum_s &: p_{s1} + p_{s2} + \dots + p_{s(k+1)} = r_s, \quad s = 1, \dots, m-2, m+1, \dots, n, \\
r &= p_1 + p_2 + \dots + p_{k+1} = r_1 + r_2 + \dots + r_n, \\
p_j &= p_{1j} + p_{2j} + \dots + p_{nj}, \quad j = 1, 2, \dots, k+1. \tag{17}
\end{aligned}$$

Especially, due to r_1, \dots, r_n are nonnegative integers, thus from the first equation of (17),

one has $r_n = 0$, which make condition (17) become

$$\begin{aligned}
& \sum_0 : r_1 + \cdots + (m-1)(r_{m-1} + 1) + m(r_m - 1) + \cdots + (n-1)r_{n-1} = n-1, \\
& \sum_{m-1} : p_{(m-1)1} + p_{(m-1)2} + \cdots + p_{(m-1)(k+1)} = r_{m-1} + 1, \quad m = 2, \dots, n-1, \\
& \sum_m : p_{m1} + p_{m2} + \cdots + p_{m(k+1)} = r_m - 1, \quad m = 2, \dots, n-1, \\
& \sum_s : p_{s1} + p_{s2} + \cdots + p_{s(k+1)} = r_s, \quad s = 1, \dots, m-2, m+1, \dots, n, \\
& r = p_1 + p_2 + \cdots + p_{k+1} = r_1 + r_2 + \cdots + r_{n-1}, \\
& p_j = p_{1j} + p_{2j} + \cdots + p_{(n-1)j}, \quad j = 1, 2, \dots, k+1.
\end{aligned} \tag{18}$$

The condition (18) is just the requirement of the parameters in $(\partial^{n-1}E_0/\partial q^{n-1})|_{q=0}$, and the coefficient of θ in Eq.(14) corresponds to $(\partial^{n-1}E_0/\partial q^{n-1})|_{q=0}$ multiplied by $n(n-1)$. Hence, we prove that Eq.(14) involves $n(n-1)(\partial^{n-1}E_0/\partial q^{n-1})|_{q=0}$.

Enlarging the degree of θ and repeating similar procedure as above up to θ^{n-2} , we find that Eq.(14) contains $\left[\sum_{i=2}^{n-1} \frac{n!}{(i-1)!} \binom{n-1}{i-2} \theta^{n-i+1} \frac{\partial^{i-1}E_0}{\partial q^{i-1}} \right]_{q=0}$, thus at last, we consider the maximal degree of θ , that is θ^{n-1} , which only occurs in the terms containing u_n or its derivatives. Hence, this case corresponds to $r = 1$ in Lemma 1, which implies the terms corresponding to θ^{n-1} in Eq.(14) reads

$$n! \left[\frac{\partial E_0}{\partial u^{(0)}|_{q=0}} u_1^{(0)} + \frac{\partial E_0}{\partial u^{(1)}|_{q=0}} u_1^{(1)} + \cdots + \frac{\partial E_0}{\partial u^{(k)}|_{q=0}} u_1^{(k)} \right], \tag{19}$$

which is just the $(\partial E_0/\partial q)|_{q=0}$ multiplied by $n!$. This prove it. \square

Next, for $\theta \neq 0$, we give the main theorem about the connections between Eq.(3) and Eq.(7).

Theorem 1. Eq.(7) is equivalent to Eq.(3) under the transformation

$$u_0^{(\lambda)} = \tilde{u}_0^{(\lambda)}, \quad u_l^{(\lambda)} = \sum_{j=0}^{l-1} \binom{l-1}{j} \theta^j [\epsilon(1-\theta)]^{l-j} \tilde{u}_{l-j}^{(\lambda)}, \tag{20}$$

where $l = 1, 2, \dots$, $\lambda = 0, 1, \dots, k$.

Proof. We start with Eq.(7) and list it for different order of q up to l in detail as follows

$$\begin{aligned}
& E_0|_{q=0} = 0, \\
& \left[\frac{\partial E_0}{\partial q} + \epsilon(1-\theta)E_1 \right]_{q=0} = 0, \\
& \left[\frac{\partial^2 E_0}{\partial q^2} + 2\epsilon(1-\theta) \left(\frac{\partial E_1}{\partial q} + \theta E_1 \right) \right]_{q=0} = 0, \\
& \left[\frac{\partial^3 E_0}{\partial q^3} + 3\epsilon(1-\theta) \left(\frac{\partial^2 E_1}{\partial q^2} + 2\theta \frac{\partial E_1}{\partial q} + 2\theta^2 E_1 \right) \right]_{q=0} = 0, \\
& \quad \dots \\
& \left[\frac{\partial^l E_0}{\partial q^l} + l\epsilon(1-\theta) \left(\frac{\partial^{l-1} E_1}{\partial q^{l-1}} + (l-1)\theta \frac{\partial^{l-2} E_1}{\partial q^{l-2}} + \cdots + (l-1)!\theta^{l-1} E_1 \right) \right]_{q=0} = 0.
\end{aligned} \tag{21}$$

The proof of Lemma 1 tells us that $\partial^l E_0 / \partial q^l$ with $q = 0$ is linear about u_l ($l = 1, 2, \dots$) and its derivatives, thus we first consider transformation (12) for Eq.(21).

The first equation in Eq.(21) is unperturbed equation which is the same as the first equation in Eq.(3). For the second equation, by Lemma 1, $(\partial E_0 / \partial q)|_{q=0}$ is linear about u_1 and its derivatives, thus one can adopt a scaling transformation to convert it to the same form as the second one in Eq.(3).

Next, we consider the third equation in Eq.(21). Take the first three transformations of (12) into the third equation in Eq.(21) and minus the second equation multiplied by 2θ , then by means of Lemma 2, one obtains

$$\left[\frac{\partial^2}{\partial q^2} E_0(x, \hat{u} \dots, \hat{u}^{(k)}) + 2\epsilon(1 - \theta) \frac{\partial}{\partial q} E_1(x, \hat{u} \dots, \hat{u}^{(k)}) \right]_{|q=0} = 0, \quad (22)$$

which has the same form as the third equation in Eq.(3)

$$\left[\frac{\partial^2 E_0}{\partial \epsilon^2} + 2 \frac{\partial E_1}{\partial \epsilon} \right]_{|\epsilon=0} = 0$$

except for the coefficient of $\partial E_1 / \partial q$.

In the same way, substituting the first four transformations of (12) into the fourth equation and minus the sum of the second equation multiplied by $6\theta^2$ and Eq.(22) multiplied by 6θ , also using Lemma 2, we have

$$\left[\frac{\partial^3}{\partial q^3} E_0(x, \hat{u} \dots, \hat{u}^{(k)}) + 3\epsilon(1 - \theta) \frac{\partial^2}{\partial q^2} E_1(x, \hat{u} \dots, \hat{u}^{(k)}) \right]_{|q=0} = 0, \quad (23)$$

which has the same form as the fourth equation in Eq.(3)

$$\left[\frac{\partial^3 E_0}{\partial \epsilon^3} + 3 \frac{\partial^2 E_1}{\partial \epsilon^2} \right]_{|\epsilon=0} = 0$$

except for the coefficient of $\partial^2 E_1 / \partial q^2$.

Enlarging l inductively and assuming that the first l equations of Eq.(21) have been transformed to

$$\left[\frac{\partial^{l-1}}{\partial q^{l-1}} E_0(x, \hat{u} \dots, \hat{u}^{(k)}) + \epsilon(l-1)(1 - \theta) \frac{\partial^{l-2}}{\partial q^{l-2}} E_1(x, \hat{u} \dots, \hat{u}^{(k)}) \right]_{|q=0} = 0, \quad (24)$$

then insert the first $(l+1)$ transformations of (12) into the $(l+1)$ th equation and minus the sum of the i th equation of the form (24) multiplied by $\frac{l!}{(i-1)!} \binom{l-1}{i-2} \theta^{l-i+1}$ ($i = 2, 3, \dots, l$), we convert the last equation of Eq.(21) to the form

$$\left[\frac{\partial^l}{\partial q^l} E_0(x, \hat{u} \dots, \hat{u}^{(k)}) + \epsilon l(1 - \theta) \frac{\partial^{l-1}}{\partial q^{l-1}} E_1(x, \hat{u} \dots, \hat{u}^{(k)}) \right]_{|q=0} = 0, \quad (25)$$

which has the same form as the $(l+1)$ th equation in Eq.(3)

$$\left[\frac{\partial^l E_0}{\partial \epsilon^l} + l \frac{\partial^{l-1} E_1}{\partial \epsilon^{l-1}} \right]_{|\epsilon=0} = 0$$

except for the coefficient of $\partial^{l-1}E_1/\partial q^{l-1}$.

At last, by Lemma 1, using the scaling transformation

$$\widehat{u}_0^{(\lambda)} = \widetilde{u}_0^{(\lambda)}, \quad \widehat{u}_1^{(\lambda)} = \epsilon(1-\theta)\widetilde{u}_1^{(\lambda)}, \quad \widehat{u}_2^{(\lambda)} = \epsilon(1-\theta)\widetilde{u}_2^{(\lambda)}, \quad \dots, \quad \widehat{u}_l^{(\lambda)} = [\epsilon(1-\theta)]^l \widetilde{u}_l^{(\lambda)}, \quad (26)$$

one can convert Eq.(25) to the $(l+1)$ th equation of Eq.(3) by ASM.

In summary, combining transformations (12) and (26), we obtain transformation (20), which connects Eq.(3) and Eq.(7). This proves it. \square

Transformation (20) in Theorem 1 build a bridge between AHSM and ASM for perturbed PDEs, that is to say, there is no need to perform AHSM to the governing equations because the desired approximate homotopy series solutions can be obtained by acting the transformation (20) on the approximate solutions by ASM.

3 Applications

In this section, we apply transformation (20) to study the connections between ASM and AHSM for the Cahn-Hilliard equation [9, 10]

$$u_t = -[F(u)u_x]_x - \epsilon u_{xxxx}, \quad (27)$$

which is one of the simplest models to characterize the process in the context of the continuum theory of phase transitions. In Eq.(27), x, t are two independent variables and u is a scalar dependent one, the nonlinearity $F(u)$ is the derivative of a double-well potential with wells of equal depth and $0 < \epsilon \ll 1$ measures the thickness of an interface separating the two preferred states of the system.

3.1 The connections between the couple equations

From Eq.(27), one has

$$E_0(u) = u_t + [F(u)u_x]_x, \quad E_1(u) = u_{xxxx}.$$

By means of the general formulas of couple system (7), we get the following couple equations by AHSM

$$u_{k,t} + \left[\sum_{i=0}^k \frac{F^{(j)}(u_0)}{j_0!j_1!\dots j_i!} u_{l_0}^{j_0} u_{l_1}^{j_1} \dots u_{l_i}^{j_i} u_{k-i,x} \right]_x + \epsilon(1-\theta) \sum_{i=0}^{k-1} \theta^{k-1-i} u_{i,xxxx} = 0. \quad (28)$$

where, hereinafter, $u_{-1} = 0$, $u_{l_i}^{j_i} = (u_{l_i})^{j_i}$, $u_{k,t} = \partial u_k / \partial t$, $u_{i,xxxx} = \partial^4 u_i / \partial x^4$, $j_0 + j_1 + \dots + j_i = j$, $l_0 j_0 + l_1 j_1 + \dots + l_i j_i = i$, $0 \leq j \leq i$ and l_0, l_1, \dots, l_i are not equal to zero and mutual inequivalent. l_i, j_i, j, k ($i, j, k = 0, 1, 2, \dots$) are nonnegative integers and satisfy the above relations.

Now, we use transformation (20) to convert Eq.(28) to the couple equations by ASM. In order to give specific procedure about how transformation (20) works for Eq.(28), we first

multiply both sides of Eq.(28) with $k!$ in order to make it with integer coefficients and then list it for different order of q in detail as follows

$$\begin{aligned}
u_{0,t} + [F(u_0)u_{0,x}]_x &= 0, \\
u_{1,t} + [F(u_0)u_1]_{xx} + \epsilon(1-\theta)u_{0,xxxx} &= 0, \\
2u_{2,t} + [F'(u_0)u_1^2 + 2F(u_0)u_2]_{xx} + 2\epsilon(1-\theta)[u_{1,xxxx} + \theta u_{0,xxxx}] &= 0, \\
6u_{3,t} + [F''(u_0)u_1^3 + 6F'(u_0)u_1u_2 + 6F(u_0)u_3]_{xx} \\
&\quad + 6\epsilon(1-\theta)[u_{2,xxxx} + \theta u_{1,xxxx} + \theta^2 u_{0,xxxx}] = 0, \\
&\dots \\
k! u_{k,t} + \left[\sum_{i=0}^k \frac{k! F^{(j)}(u_0)}{j_0! j_1! \dots j_i!} u_{l_0}^{j_0} u_{l_1}^{j_1} \dots u_{l_i}^{j_i} u_{k-i,x} \right]_x + \epsilon(1-\theta) k! \sum_{i=0}^{k-1} \theta^{k-1-i} u_{i,xxxx} &= 0.
\end{aligned} \tag{29}$$

Like the proof of Theorem 1, we first consider transformation

$$u_0^{(\lambda)} = \hat{u}_0, \quad u_k^{(\lambda)} = \sum_{i=0}^k \binom{k-1}{i} \theta^i \hat{u}_{k-i}^{(\lambda)}, \quad k = 1, 2, \dots, \lambda = 0, 1, \dots, k. \tag{30}$$

By means of Theorem 1, the first two equations of Eq.(29) are equivalent to the corresponding ones by ASM under a scaling transformation $u_0 = \hat{u}_0, u_1 = \epsilon(1-\theta)\hat{u}_1$, thus we consider the third equation. Take the first three transformations of (30) into the third equation and minus the second equation multiplied by 2θ , one obtains

$$2\hat{u}_{2,t} + [F'(\hat{u}_0)\hat{u}_1^2 + 2\hat{u}_2 F(\hat{u}_0)]_{xx} + 2\epsilon(1-\theta)\hat{u}_{1,xxxx} = 0. \tag{31}$$

Similarly, substitute the first four transformations of (30) into the fourth equation and minus the sum of the second equation multiplied by $6\theta^2$ and Eq.(31) multiplied by 6θ , we have

$$6\hat{u}_{3,t} + [F''(\hat{u}_0)\hat{u}_1^3 + 6\hat{u}_2\hat{u}_1 F'(\hat{u}_0) + 6\hat{u}_3 F(\hat{u}_0)]_{xx} + 6\epsilon(1-\theta)\hat{u}_{2,xxxx} = 0.$$

Suppose the first k equations in Eq.(29) be in the form

$$\begin{aligned}
(k-1)! u_{k-1,t} + \left[\sum_{i=0}^{k-1} \frac{(k-1)! F^{(j)}(u_0)}{j_0! j_1! \dots j_i!} u_{l_0}^{j_0} u_{l_1}^{j_1} \dots u_{l_i}^{j_i} u_{k-1-i,x} \right]_x \\
+ \epsilon(1-\theta)(k-1)! u_{k-2,xxxx} = 0,
\end{aligned} \tag{32}$$

then insert the first $(k+1)$ transformations into the $(k+1)$ th equation and minus the sum of the i th equation of the form (32) multiplied by $\frac{k!}{(i-1)!} \binom{k-1}{i-2} \theta^{k-i+1}$ ($i = 2, 3, \dots, k$), we convert the last equation of Eq.(29) to the form

$$k! u_{k,t} + \left[\sum_{i=0}^k \frac{k! F^{(j)}(u_0)}{j_0! j_1! \dots j_i!} u_{l_0}^{j_0} u_{l_1}^{j_1} \dots u_{l_i}^{j_i} u_{k-i,x} \right]_x + \epsilon(1-\theta) k! u_{k-1,xxxx} = 0.$$

Finally, since the i th equation of Eq.(29) is linear about u_i and its derivatives ($i = 1, 2, \dots, k$), thus integrating the scaling transformation

$$\hat{u}_0^{(\lambda)} = \tilde{u}_0^{(\lambda)}, \quad \hat{u}_1^{(\lambda)} = \epsilon(1-\theta)\tilde{u}_1^{(\lambda)}, \quad \hat{u}_2^{(\lambda)} = \epsilon(1-\theta)\tilde{u}_2^{(\lambda)}, \quad \dots, \quad \hat{u}_k^{(\lambda)} = [\epsilon(1-\theta)]^k \tilde{u}_k^{(\lambda)},$$

and (30), one obtains (20), which convert Eq.(28) to the form

$$k! u_{k,t} + \left[\sum_{i=0}^k \frac{k! F^{(j)}(u_0)}{j_0! j_1! \dots j_i!} u_{l_0}^{j_0} u_{l_1}^{j_1} \dots u_{l_i}^{j_i} u_{k-i,x} \right]_x + k! u_{k-1,xxxx} = 0,$$

after deleting the symbol \sim , which is just the couple equations via formulas (3) by ASM.

3.2 Homotopy series solutions through the transformation

In this subsection, we use transformation (20) to obtain approximate homotopy series solutions from the approximate solutions by ASM.

I. Traveling wave form

By ASM, the traveling wave solution with third-order precision for $F(u) = 1/u$ is given by $u = \tilde{u}_0 + \epsilon \tilde{u}_1 + \epsilon^2 \tilde{u}_2 + \epsilon^3 \tilde{u}_3$ after selecting proper parameters, where

$$\tilde{u}_0 = \frac{1}{a(at-x)}, \tilde{u}_1 = -\frac{3}{a^2(at-x)^4}, \tilde{u}_2 = \frac{774}{10a^3(at-x)^7}, \tilde{u}_3 = -\frac{51273}{10a^4(at-x)^{10}},$$

and a is wave speed.

By virtue of the first four transformations of (20), i.e.,

$$\begin{aligned} u_0 &= \tilde{u}_0, u_1 = \epsilon(1-\theta)\tilde{u}_1, \\ u_2 &= \epsilon(1-\theta)[\epsilon(1-\theta)\tilde{u}_2 + \theta\tilde{u}_1], \\ u_3 &= \epsilon(1-\theta)[\epsilon^2(1-\theta)^2\tilde{u}_3 + 2\theta\epsilon(1-\theta)\tilde{u}_2 + \theta^2\tilde{u}_1], \end{aligned}$$

we obtain a third-order approximate homotopy series solution of Eq.(27) in the form $u = u_0 + qu_1 + q^2u_2 + q^3u_3$, which is

$$\begin{aligned} u &= \frac{1}{a^2t - ax} + \frac{3\epsilon q(\theta-1)}{a^2(x-at)^4} + \frac{3\epsilon q^2(\theta-1)[5a\theta(at-x)^3 + 129\epsilon(\theta-1)]}{5a^3(at-x)^7} \\ &+ \frac{3\epsilon q^3(\theta-1)[10a^2\theta^2(x-at)^6 + 516\epsilon\theta a(\theta-1)(at-x)^3 + 17091\epsilon^2(\theta-1)^2]}{10a^4(x-at)^{10}}. \end{aligned} \quad (33)$$

Inserting (33) into Eq.(27) and putting $x = 1, t = 0.1, \epsilon = 0.01$, we get figure 1 which demonstrates the error of solution (33) with respect to the auxiliary parameter θ .

From Figure 1, in particular, assuming convergence-control parameter $\theta = 0.5478$, one has

$$u_t + [F(u)u_x]_x + \epsilon u_{xxxx} \approx 4.70 \times 10^{-6}.$$

II. $F = u$

We construct the following approximate solutions with fourth-order precision after assigning proper value to some parameters by ASM

$$u = \frac{x^2}{6t} + \frac{\epsilon}{x^2} - \frac{30\epsilon^2 t}{x^6} - \frac{46440\epsilon^3 t^2}{7x^{10}} - \frac{804646440\epsilon^4 t^3}{203x^{14}}.$$

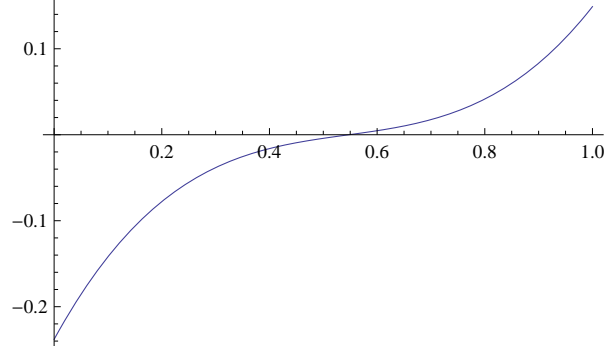


Figure 1: $x = 1, t = 0.1, \epsilon = 0.01$, horizontal axis denote θ , vertical axis denote the error

By virtue of the first five transformations of (20), a fourth-order approximate homotopy series solution follows

$$\begin{aligned}
 u = & \frac{x^2}{6t} - \frac{\epsilon q(\theta - 1)}{x^2} - \frac{\epsilon q^2(\theta - 1)(390\epsilon(\theta - 1)t + 13\theta x^4)}{13x^6} \\
 & + \frac{\epsilon q^3(\theta - 1)(603720\epsilon^2(\theta - 1)^2 t^2 - 5460\epsilon\theta(\theta - 1)tx^4 - 91\theta^2 x^8)}{91x^{10}} \\
 & + \epsilon q^4(\theta - 1) \left[\frac{-10460403720\epsilon^3(\theta - 1)^3 t^3 + 52523640\epsilon^2\theta(\theta - 1)^2 t^2 x^4}{2639x^{14}} \right. \\
 & \left. - \frac{237510\epsilon\theta^2(\theta - 1)tx^8 + 2639\theta^3 x^{12}}{2639x^{14}} \right]. \tag{34}
 \end{aligned}$$

When the constants and variables in solution (34) are specified by $x = 1, t = 0.1, \epsilon = 0.01$, the relationship between the error of fourth-order precision approximate homotopy series solutions of (34) and parameter θ is sketched in Figure 2.

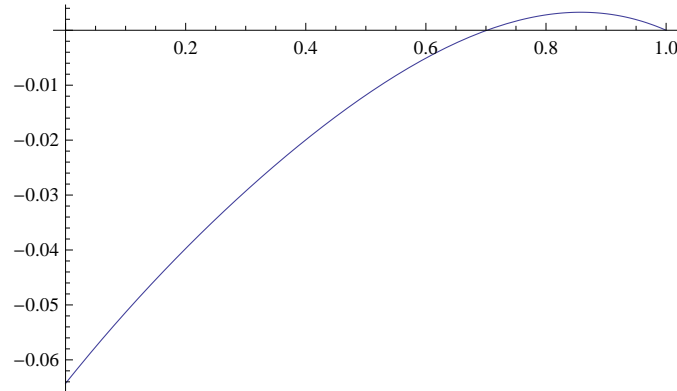


Figure 2: $x = 1, t = 0.1, \epsilon = 0.01$, horizontal axis denote θ , vertical axis denote the error

Especially, taking convergence-control parameter $\theta = 0.7015$, one has

$$u_t + [F(u)u_x]_x + \epsilon u_{xxxx} \approx 1.59 \times 10^{-6}.$$

4 Conclusion and discussion

In this paper, we show that the two couple equations derived by ASM and AHSM are connected by the transformation (20). As a result, approximate homotopy series solutions can be obtained by acting the transformation on the known solutions by ASM. As an application, the Cahn-Hilliard equation is considered to examine the results. This transformation may exert an important role in studying the perturbed PDEs because it makes researchers facilitate to study the results by AHSM through the ones by ASM.

In addition, it is worthwhile pointing out that, for Eq.(27), the unique pair of different operators Y_1, X_1 , i.e., the approximate homotopy symmetry operator

$$Y_1 = -\frac{1}{2}x\partial_x - t\partial_t + \sum_{i=1}^{\infty} [iu_i - (i-1)\theta u_{i-1}] \partial_{u_i}$$

and the corresponding approximate symmetry one

$$X_1 = -\frac{1}{2}x\partial_x - t\partial_t + \sum_{i=0}^{\infty} i\tilde{u}_i \partial_{\tilde{u}_i}$$

are equivalent under the transformation

$$u_0 = \tilde{u}_0, \quad u_l = \sum_{i=0}^{l-1} \binom{l-1}{i} \frac{l-i}{l} \theta^i \tilde{u}_{l-i}, \quad l = 1, 2, \dots,$$

other than (20). Hence, the relations between the symmetries of two equivalent equations may form an interesting topic.

References

- [1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York 1993.
- [2] G.W. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer, New York 2004.
- [3] V.A. Baikov, R.K. Gazizov, N.H. Ibragimov, Perturbation methods in group analysis, J. Soviet. Math. 55 (1991) 1450-1490.
- [4] W.I. Fushchich, W.H. Shtelen, On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter, J. Phys. A: Math. Gen. 22 (1989) L887-L890 .
- [5] M. Pakdemirli, M. Yurusoy, T. Dolapci, Comparison of Approximate Symmetry Methods for Differential Equations, Acta Appl. Math. 80 (2004) 243-271.
- [6] R. Wiltshire, Two approaches to the calculation of approximate symmetry exemplified using a system of advection-diffusion equations, J. Comput. Appl. Math. 197 (2006) 287-301.

- [7] X.Y. Jiao, Y. Gao, S.Y. Lou, Approximate homotopy symmetry method: Homotopy series solutions to the sixth-order Boussinesq equation, *Sci. China. Ser. G-Phys. Mech. Astron.* 52 (2009) 1169-1178.
- [8] Z.Y. Zhang, Y.F. Chen, A comparative study of approximate symmetry and approximate homotopy symmetry to a class of perturbed nonlinear wave equation, *Nonlinear Anal.* 74 (2011) 4300-4318.
- [9] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* 28 (1958) 258-267.
- [10] X.D. Sun, M. J. Ward, Dynamics and Coarsening of Interfaces for the Viscous Cahn-Hilliard Equation in One Spatial Dimension, *Stud. Appl. Math.* 105 (2000) 203-234.
- [11] R.L. Mishkov, Generalization of the formula of Fa di Bruno for a composite function with a vector argument, *Internat. J. Math.& Math. Sci.* 24 (2000) 481-491.